Matrix groups as manifolds

- **Def. 1: Smooth embedded submanifolds of $\mathbb{R}^n$**

Let $0 \leq m \leq n$, $m, n \in \mathbb{N}$.
A C^∞-embedded submanifold of $\mathbb{R}^n$ of dim. m is a non-empty set $M \subseteq \mathbb{R}^n$ such that $\forall x_0 \in M$

- $\exists\, \Omega \subseteq \mathbb{R}^n$ open, $x_0 \in \Omega$
- $\exists\, U \subseteq \mathbb{R}^m$ open
- $\exists\, \phi \in C^\infty(U, \mathbb{R}^n)$ and $D\phi(x_0)$ is injective

such that $M \cap \Omega = \phi(U)$ and $\phi: U \to M \cap \Omega$ is a homeomorphism.
The pair (U, ϕ) is called a chart.
The set of all charts is called an atlas.

- **Def. 2: Tangentspace**

Let $M \subseteq \mathbb{R}^n$ be a m-dim smooth submanifold and $x \in M$.
The vector $v \in \mathbb{R}^n$ is called a tangent vector to M at the point x, if there is a function $\gamma \in C^1((-\varepsilon, \varepsilon), M)$, $\varepsilon > 0$, such that $\gamma(0) = x$ and $\gamma'(0) = v$.
The set of all tangent vectors $T_x M \subseteq \mathbb{R}^n$ to M at x is called the tangent space (m-dim. VS).

Comment 1: About differentiability of functions
For the Euclidean case differentiability of a function $f: M \to N$, where $M \subseteq \mathbb{R}^m$, $N \subseteq \mathbb{R}^n$ are smooth embedded submanifolds is canonically defined.
Notice that in the general topological case the concept is (a priori) not defined in any way.

- **Def. 3: Lie-group**

Let G be a smooth manifold which is also a group
Let $mult: G \times G \to G$, $inv: G \to G$.
Then G is called a Lie group if mult and inv are smooth maps.

- Examples of smooth embedded submanifolds:
  - $S^{n-1} \subseteq \mathbb{R}^n$ is a $(n-1)$-dim. submanifold
  - $\mathbb{R}^n$ is n-dim submanifold
  - every open subset of $\mathbb{R}^n$ is n-dim. submanifold

- Examples of Lie groups:
  - Isometries: $O(n)$, $U(n)$, $SO(n)$, $SU(n)$
  - $\mathbb{R}^n$ with vector addition
  - $GL_n\mathbb{K}$, $SL_n\mathbb{K}$

- **Theorem 1: Matrix groups and Lie groups**

Every matrix group is a Lie group.
The inverse implication is in general wrong.